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Concerning Systems of Conics Lying on Cubic Quartic and Quintic Surfaces.

By C. H. Sisam.

INTRODUCTION.

Although the properties of algebraic ruled surfaces have been extensively studied, and the classification of such surfaces through the sixth, and, in some cases, for higher orders, has been exhaustively carried out; yet, except for certain surfaces generated by circles, such as surfaces of revolution and annular surfaces, and for surfaces containing a doubly infinite system of conics; i. e., the Steiner surfaces, the properties of surfaces generated by systems of conics has received little consideration. To determine some of the properties of certain of those surfaces and of the systems of conics lying on them is the object of this paper.

Among the leading articles dealing with this subject which have appeared, I may mention Koenigs'* paper on surfaces multiply generated by conics, also Stuyvaert's† paper on the properties of systems of conics determined by the condition of intersecting given fixed curves. Bertini‡ and Nugteren§ have considered special cases of the problem considered by Stuyvaert. Emil Weyr‖ has considered the problem of constructing the tangent planes to a surface along an arbitrary conic of a system lying on it.

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* "Determination de toutes les surfaces plusieurs fois engendrées par des coniques." Annales de L'Ecole Normale Superieure, Series 3, No. V.

† "Etude de quelques surfaces algébriques engendrées par des courbes du second et du troisième ordre." Dissertation, Gand, 1902.

‡ "Sulle curve gobbe razionali del quinto ordine" in the Collectanea Mathematica in memoriam D. Chelini Mediolani, 1881, pp. 313–326.

§ "Rationale Ruimtekrommen van de fijde Orde." Dissertation, Utrecht, 1902.

‖ "Zur Theorie der Flächen, welche eine Schaar von Kegelschnitten enthalten." Monatshefte für Mathematik und Physik, Vol. II.

I. *On the system of tangent planes along a conic.*

The parametric equations of any surface containing a system of conics can be put in the form:

$$\chi_i = \theta_i(u) + 2v\phi_i(u) + v^2\psi_i(u) \qquad i = 1, 2, 3, 4.$$

The tangent plane to the surface at any point (u, v) is:

$$\begin{vmatrix} \chi_1 & \chi_2 & \chi_3 & \chi_4 \\ \theta_1 + v\phi_1 & \theta_2 + v\phi_2 & \theta_3 + v\phi_3 & \theta_4 + v\phi_4 \\ \phi_1 + v\psi_1 & \phi_2 + v\psi_2 & \phi_3 + v\psi_3 & \phi_4 + v\psi_4 \\ \theta_1' + 2v\phi_1' + v^2\psi_1' & \theta_2' + 2v\phi_2' + v^2\psi_2' & \theta_3' + 2v\phi_3' + v^2\psi_3' & \theta_4' + 2v\phi_4' + v^2\psi_4' \end{vmatrix} = 0,$$

in which θ_i', ϕ_i' and ψ_i' denote derivatives with respect to u.

This equation is of fourth degree in v. Hence, in general, the tangents to the surface along a fixed conic $u =$ const. form a developable of class four. This developable has the plane of the given conic for double plane. For it is easily seen that, at each of the points of intersection of the given conic with the plane of the consecutive conic $u + du$, the plane of the given conic is tangent to the surface.

If, however, for all values of u, this developable reduces to one of class three, then for some value of v the minors of $\chi_1\ \chi_2\ \chi_3$ and χ_4 in the above determinant must all vanish. This value of v is, in general, a function of u, say $v = f(u)$, but on replacing v by $v + f(u)$, we may, without altering the *form* of the equations of the surface, reduce this value to $v = 0$. Suppose this done. It then follows that

$$\left\| \begin{matrix} \theta_1 & \theta_2 & \theta_3 & \theta_4 \\ \phi_1 & \phi_2 & \phi_3 & \phi_4 \\ \theta_1' & \theta_2' & \theta_3' & \theta_4' \end{matrix} \right\| \equiv 0$$

for all values of u.

It is thus seen that the surface belongs to one or the other of two classes:

a_1. The conics all pass through a fixed point.
a_2. The conics all touch a fixed curve.

The equations of a surface of the kind a_2 may be written in the form:

$$\chi_i = \theta_i + 2v\theta_i' + v^2\psi_i. \qquad i = 1, 2, 3, 4.$$

The analogy of the surfaces a_1 to cones and of the surfaces a_2 to developable surfaces is at once evident.

To the curve:

$$\chi_i = \theta_i(u) \qquad i = 1, 2, 3, 4$$

which is touched by all the conics of the system on a surface a_2. Darboux has given the name of "edge of regression." The analogy to the edge of regression of a developable is obvious. It is, in general, a double curve on the surface. Indeed, the only surfaces on which it is not nodal are easily seen to be those through each point of which pass two conics of the system. Koenigs[*] has shown that the only surfaces through every point of which pass two conics of the system are those which contain a doubly infinite system of conics; i. e., the Steiner surface and its degenerate cases, the ruled cubic, quadric and plane.

In the determinant equation of the tangent plane, the minors of χ_1, χ_2, χ_3 and χ_4 may all contain a common factor quadratic in v. The tangent planes along an arbitrary conic then envelope a quadric cone. This happens when the surface belongs to one of the following classes:

b_1. The conics all pass through two fixed points.

b_2. The conics all pass through a fixed point and touch a fixed curve.

b_3. The conics all touch a fixed curve (which may be either proper or composite) at two points.[†]

b_4. The conics all have contact of the second order with a fixed curve. In the case of the surfaces b_4 the quadratic factor common to the four minors is the square of a linear factor. The equations of such a surface may be written in the form:

$$\chi_i = \theta_i + v\,\theta_i' + v^2\left(\frac{\theta_i''}{2} + a\,\theta_i' + b\,\theta_i\right) \qquad i = 1, 2, 3, 4$$

a and b being constants.

All the conics have three point contact with the curve:

$$\chi_i = \theta_i(u) \qquad i = 1, 2, 3, 4$$

This curve is, in general, triple on the surface; at each of its points the three tangent planes coincide. No surface of this kind is of order low as five. A simple example of such a surface is:

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[*] Loc. cit.

[†] Enneper writing in the Zeitschrift für Mathematik und Physik, in 1869, and, following him, Cosserat in the Annales des Faculté des Sciences de Toulouse, in 1889, have inferred that the tangents along a conic may envelope a quadric cone when the given conic meets the consecutive conic at only one of its intersections with the plane of the latter. Since, however, the plane of the given conic would have to be tangent to the surface at the second intersection, the developable of tangents would have to be of third class.

$$\chi_1(\chi_3^2-\chi_2\chi_4)^3+3\chi_2(\chi_3^2-\chi_2\chi_4)^2(\chi_2^2-\chi_1\chi_3)+3\chi_3(\chi_3^2-\chi_2\chi_4)(\chi_2^2-\chi_1\chi_3)^2$$
$$+\chi_4(\chi_2^2-\chi_1\chi_3)^3=0$$

on which the twisted cubic:

$$\chi_3^2-\chi_2\chi_4=0 \qquad \chi_2^2-\chi_1\chi_3=0$$

has three point contact at each of its points with a conic in the plane:

$$\chi_1 u^3+3\chi_2 u^2+3\chi_3 u+\chi_4=0.$$

Since a conic is not completely determined by the conditions of having contact of first or second order with a given curve at a given point, the curve

$$\chi_i=\theta_i(u) \qquad\qquad i=1,\,2,\,3,\,4$$

may have contact of the first or second order at each of its points with several conics of the system. Thus, the line $\chi_1=\chi_2=0$ on the surface,

$$\chi_1^4\chi_3^2+\chi_1^4\chi_2\chi_3+2\chi_1^2\chi_2^2\chi_3\chi_4+\chi_1\chi_2^4\chi_4+\chi_2^4\chi_4^2=0,$$

is touched at each of its points by two conics of the system lying in the pencil of planes through the line.

Similarly, on the surface:

$$\chi_1(\chi_3^2-\chi_2\chi_4)^6+3\chi_2(\chi_3^2-\chi_2\chi_4)^4(\chi_2^2-\chi_1\chi_3)^2+3\chi_3(\chi_3^2-\chi_2\chi_4)^2(\chi_2^2-\chi_1\chi_3)^4$$
$$+\chi_4(\chi_2^2-\chi_1\chi_3)^6=0,$$

the twisted cubic:

$$\chi_3^2-\chi_2\chi_4=0 \qquad \chi_2^2-\chi_1\chi_3=0$$

has contact of the second order at each of its points with two conics belonging to the same system.

It will presently be shown that, in general, on a surface generated by conics there are points through which pass two consecutive conics. These points may, by analogy with the corresponding singularity on ruled surfaces, be called pinch-points. As in the case of ruled surfaces these points are uniplanar points. Along a conic passing through a pinch-point, the developable of tangents reduces to class three. The condition for such a conic is, therefore, that, for a particular value of u, the determinant given on page 100 reduce to one of third degree in v.

There exist, also, surfaces on which discrete conics meet the conics consecutive to them in two points or which meet two consecutive conics in a uniplanar triple point. The tangents to the surface along such a conic envelope a quadric cone.

Finally, two (or more) consecutive conics may be coplanar. The plane of such a conic touches the surface all along the conic.

II. *On the determination of the properties of surfaces containing unicursal systems of conics.*

In the equations

$$\chi_i = \theta_i(u) + 2v\,\phi_i(u) + v^2\,\psi_i(u) \qquad i = 1, 2, 3, 4$$

the expressions $\theta_i(u)$, $\phi_i(u)$ and $\psi_i(u)$ may, when the system is unicursal, be taken to be polynomials in u. The conics therefore lie in the planes of a developable whose equations may be put in the form:

$$L_1 u^m + m L_2 u^{m-1} + \frac{m(m-1)}{2} L_3 u^{m-2} + \ldots\ldots\ldots + L_{m+1} = 0$$

in which — as throughout this paper — $L_i = 0$ is the equation of a plane. They also lie on the surfaces of a system of quadrics of the form:

$$Q_1 u^n + n\,Q_2 u^{n-1} + \frac{n(n-1)}{2} Q_3 u^{n-2} + \ldots\ldots\ldots + Q_{n+1} = 0,$$

$Q_i = 0$ being the equation of a quadric surface.

Any conic of the system is the intersection of the plane and quadric determined by the same value of u. The surface on which the sytem lies is found by eliminating u between the two equations. Its order, M, is, in general, $2m + n$. When, however, any plane is a component of its corresponding quadric the order of the surface is reduced. If, in this case, $n \geqq m$, then the system of quadrics can be replaced by one of lower degree in u. For, let $u = 0$ be the value of u for which the plane is a component of the quadric. This is no further restriction on the system. We then have:

$$L_1 u^m + m L_2 u^{m-1} + \ldots\ldots\ldots + L_{m+1} = 0$$
$$Q_1 u^n + n\,Q_2 u^{n-1} + \ldots\ldots\ldots + L_{m+1} L' = 0.$$

On multiplying the first equation by L' and subtracting, then dividing the resulting equation by u, the required system is obtained. We may, therefore, without loss of generality, suppone one or the other of the relations

$$M = 2m + n \quad \text{or} \quad n < m$$

satisfied.

At any point of the nodal curve of the surface defined by the system, there are two values of u for which both equations of the system are satisfied. The nodal curve is therefore determined by the conditions that those equations have two common solutions.

At a pinchpoint the two common solutions of the given equations are equal. At such a point, therefore, in addition to the equations defining the system, we have:

$$L_1 u^{m-1} + (m-1) L_2 u^{m-2} + \dots\dots + L_m = 0$$
$$Q_1 u^{n-1} + (n-1) Q_2 u^{n-2} + \dots\dots + Q_n = 0.$$

Since this is only four conditions on u and the three ratios of the coordinates of a point, there exist on the surface, in general, a finite number of pinchpoints. When a curve is enveloped by the conics of the system every point of it is a pinchpoint. The edge of regression, when it exists, is, therefore, determined by these four equations.

When, at any point, the above four equations are satisfied and also

$$L_1 u^{m-2} + (m-2) L_2 u^{m-3} + \dots\dots + L_{m-1} = 0$$
$$Q_1 u^{n-2} + (n-2) Q_2 u^{n-3} + \dots\dots + Q_{n-1} = 0$$

then three consecutive conics meet at that point. When these six equations are satisfied at every point of a curve, then that curve has contact of the second order at each of its points with the conics of the system.

The condition that two consecutive conics be coplanar, is that, for some values of u, the planes:

$$L_1 u^m + m L_2 u^{m-1} + \dots\dots + L_{m+1} = 0$$
$$L_1 u^{m-1} + (m-1) L_2 u^{m-2} + \dots\dots + L_m = 0$$

be identical.

Three consecutive conics will be coplanar if, in addition:

$$L_1 u^{m-2} + (m-2) L_2 u^{m-3} + \dots\dots + L_{m-1} = 0$$

is identical with each of the other two; and similarly for any number of consecutive coplanar conics.

There may exist on the surface certain nodal straight lines which are not determined by the condition that the equations of the system have two common solutions. Let, for example:

$$Q_{n+1} \equiv L'^2 + L_{m+1} L''$$
$$Q_n \equiv L_{m+1} L''' + L' L^{IV} + L_m L''.$$

Then $L' = L_{m+1} = 0$ is a double line on the surface, although at an arbitrary point upon it the equations of the system have only one common solution.

a. *Systems of conics on cubic surfaces.*

The residual intersection of the plane of any conic with the surface is a straight line. Similarly, an arbitrary plane through an arbitrary straight line on the surface meets the surface in a conic. When the surface is not ruled, therefore, there exist on it twenty-seven systems of conics in the planes through the twenty-seven lines. The equations of any one of these systems may be written in the form:

$$L_1 u + L_2 = 0$$
$$Q_1 u + Q_2 = 0.$$

The equation of the surface is then:

$$L_1 Q_2 - L_2 Q_1 = 0.$$

When the line $L_1 = L_2 = 0$ meets the twisted quartic curve $Q_1 = Q_2 = 0$, all the conics pass through a fixed point. This point is then a node on the surface. Conversely, the conics in the pencil of planes on any line through a node all pass through the node. It follows that the class of the developable of tangent planes along any conic is reduced by unity for every node through which the conic passes.

When the cubic is ruled, an arbitrary tangent plane meets it in a rectilinear generator and a conic. There exist, therefore, on the surface, a double infinity (∞^2) of conics; namely, those in the double infinity of tangent planes. The planes of any developable of tangents to the surface cut from the surface a system of conics. This system of conics is of the same genus as the developable, for the conics of the system are in one to one correspondence with the planes of the developable. On the ruled cubic, therefore, there exist systems of conics of every genus, whereas, on the unruled cubic, the only possible systems are unicursal.

The tangent planes to the ruled cubic along any conic lying on it form a developable of class three. All the conics lying in the planes of any developable of tangents and which do not all pass through a fixed point must, therefore, touch a fixed curve. It will be shown in the case of the Steiner surface, of which the ruled cubic is a particular case, that any curve whatever on the surface is touched by a system of conics provided only that the parametrically corresponding curve is the envelope of a system of lines.

There are only two systems of conics on the ruled cubic along which the tangents to the surface envelope a quadric cone. These are the systems through the torsal generators. The conics of each system have two consecutive common points at the pinchpoints.

b. *Unicursal systems lying on quartic surfaces.*

1. *Developable of planes form a linear pencil.*

The equations of the system are:

$$L_1 u + L_2 = 0,$$
$$Q_1 u^2 + 2 Q_2 u + Q_3 = 0.$$

The equation of the surface is, therefore:

$$Q_1 L_2^2 - 2 Q_2 L_1 L_2 + Q_3 L_1^2 = 0.$$

The nodal curve is the line $L_1 = L_2 = 0$. The surface may also have one or two additional nodal lines obtained by the method shown on page 104. These lines necessarily meet $L_1 = L_2 = 0$. When the surface has one additional nodal line it is a special case of the quartic surfaces having a nodal conic and therefore also has on it systems of conics whose planes envelope quadric cones. When the surface has two additional nodal lines, it is either ruled or a Steiner surface according as these two lines do not or do intersect.

The four intersections of the line $L_1 = L_2 = 0$ with the surface $Q_2^2 - Q_1 Q_3 = 0$ are the pinchpoints of the system. In the cases where the surface contains other systems of conics, these other systems may determine pinchpoints through which do not pass two consecutive conics of this system.

All the conics of the system may touch the line $L_1 = L_2 = 0$. This happens when, for all values of u, the quartic curve

$$Q_1 u + Q_2 = 0$$
$$Q_2 u + Q_3 = 0$$

meets that line. The line $L_1 = L_2 = 0$ is then the edge of regression of the surface for this system of conics.

2. *The planes of the conics envelope a quadric cone.*

The equations are of the form:

$$L_1 u^2 + 2 L_2 u + L_3 = 0$$
$$L_1 L_4 u + Q_2 = 0.$$

The nodal curve is the intersection of the surfaces:

$$L_4 = 0 \qquad Q_2$$

It is either a proper conic or two intersecting straight lines, either distinct or consecutive. Conversely, any quartic surface whose complete nodal curve is of any of these three kinds has on it a system of conics of the above form. When however, the nodal lines are skew, either distinct or consecutive, the surface is ruled and has on it no system of conics.

The surface may have one additional nodal line. It is then ruled or a Steiner surface according as the nodal conic is not, or is, composite.

Whenever the nodal conic is not composite it may be projected into the absolute. The surface is then a *cyclide.* All of these surfaces, therefore, whose nodal conics are not composite are projections of the cyclides.

The pinchpoints of the system are the four intersections of the nodal conic $L_4 = Q_2 = 0$ with the cone $L_2^2 - L_1 L_3 = 0$. The surfaces for which the nodal conic is the edge of regression of the system may be determined by putting:

$$Q_2 \equiv L_2^2 - L_1 L_3 + L_4 L_5 .$$

When $L_1 = 0$, $L_2 = 0$ and $L_3 = 0$ all contain the same line, the developable of the planes of the conics reduces to a linear pencil counted twice. Two conics of the system are here coplanar with the conics consecutive to them. The equations of such a system may be reduced to the form:

$$L_1 u^2 + L_2 = 0$$
$$L_1 L_3 u + Q_2 = 0.$$

All the conics of the system pass through the two points $L_1 = L_2 = Q_2 = 0$. Conversely, if two conics of any system on a quartic which is not a Steiner surface lie in the same plane, then all the conics pass through two fixed points and, if the system is unicursal, its equations can be put in the above form. When the surface is a Steiner surface, however, this theorem is not true, since two conics of the same system may pass through every point of such a surface.

When the planes of the conics form a cubic developable the surface is either a ruled quartic or a Steiner surface, since the nodal curve is a cubic.

3. *Systems of conics on ruled quartics.*

The ruled quartic is the ruled surface of highest degree having on it a system of conics. Both the system and the surface must be rational.

The planes of the system must either form a linear pencil, or touch a quadric cone or form a developable of class three. In the first case the surface has two nodal rectilinear directrices and the axis of the pencil is a double generator. In the second case the nodal curve is a rectilinear directrix and a proper conic. In the third case the nodal curve is either a triple rectilinear directrix or a double cubic.

The conics can touch a curve on the surface in two cases only; first, when their planes form a linear pencil about a cuspidal generator and, second, when the nodal curve is a proper cubic and the surface is itself developable.

4. *Systems of conics on the Steiner surface.*

The intersection of any tangent plane with the surface is a quadrinodal quartic and therefore breaks up into two conics. The developable of tangents to the surface along any such conic can not be of more than third class; for if it were of fourth class the plane of the conic would have to be a double tangent plane to the surface. The conics of any system, therefore, which is chosen so that the conics do not all pass through a fixed point, all touch a fixed curve.

It is well known that the equations of the surface can be put into the form:

$$\chi_i = a_i + b_i u + c_i v + d_i u^2 + e_i u v + f_i v^2 \qquad i = 1, 2, 3, 4$$

To the points of any line:

$$\alpha u + \beta v + \gamma = 0$$

in the (u, v) plane correspond the points of a conic on the surface. To any curve on the surface corresponds another curve

$$F(u, v) = 0.$$

The system of tangents to $F = 0$ determine on the surface a system of conics touching the corresponding curve. Hence, any curve on the surface such that the corresponding curve in the (u, v) plane is the envelope of a system of lines is itself the envelope of a system of conics.

No curve on the surface is either osculated or touched twice by a system of conics since the corresponding curve $F = 0$ can not be osculated or touched twice by a system of lines.

c. *Quintic surfaces.*

1. *The planes of the conics form a linear pencil.*

The equations of the system are of the form:

$$L_1 u + L_2 = 0,$$
$$Q_1 u^3 + Q_2 u^2 + Q_3 u + Q_4 = 0.$$

The nodal curve is the triple line $L_1 = L_2 = 0$. The surface may have one or two additional nodal lines under the conditions mentioned on page 104. It can not have three double lines because a quintic surface with a nodal curve of order six must be ruled* and a ruled quintic can not have on it a family of proper conics. When the surface has two double lines it contains another system of conics, whose planes form a developable of class three. The double lines may, in particular, be consecutive with each other or with the triple line.

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* See Picard in Crelle's Journal, Vol. 100.

The eight pinchpoints of the system are the intersections of the line $L_1 = L_2 = 0$ with the envelope of the system of quadrics. All the conics of the system will touch the triple line when, for all values of u, the quartic curves:

$$Q_1 u^2 + 2 Q_2 u + Q_3 = 0$$
$$Q_2 u^2 + 2 Q_3 u + Q_4 = 0$$

meet that line. Two tangent planes will then coincide at each point of the triple line, the third being torsal along the line.

2. *The planes of the conics envelope a quadric cone.*

The equations of the system are:

$$L_1 u^2 + 2 L_2 u + L_3 = 0,$$
$$Q_1 u + Q_2 = 0.$$

The nodal curve is the (proper or composite) quartic curve $Q_1 = Q_2 = 0$. The surface may also contain an additional nodal line under the conditions mentioned on page 104.

The eight pinchpoints of the system are the intersections of the nodal quartic with the cone

$$L_2^2 - L_1 L_3 = 0.$$

Any curve which is enveloped by all the conics of the system lies on each of the quadrics:

$$Q_1 = 0, \qquad Q_2 = 0, \qquad L_2^2 - L_1 L_3 = 0.$$

The component so enveloped may be either a conic or a cubic or a quartic curve.

The nodal curve may also be touched twice by all the conics of the system. When this happens we may, without restricting the surface, put $Q_1 \equiv L_2^2 - L_1 L_3$, Q_2 being arbitrary. The curve thus enveloped is, in general, a quartic; but it may break up into two conics each of which is touched once.

As in the corresponding case of quartic surfaces, the system of planes of the conics may degenerate into a linear pencil, each plane of which contains two conics of the system. Two conics of such a system are coplanar with the conics consecutive to them and the equations of the system may be put into the form

$$L_1 u^2 + L_2 = 0,$$
$$Q_1 u + Q_2 = 0.$$

3. *Developable of the third class.*

It will be convenient to distinguish two cases. First, let

$$L_1 u^3 + 3 L_2 u^2 + 3 L_3 u + L_4 = 0,$$
$$L_1 L_5 u + L_4 L_6 = 0.$$

(The case in which the family of quadrics is of the form:

$$(3 L_3 L_5 - L_4 L_6) u + L_4 L_5 = 0$$

is a sub-case of the above.)

The line $L_5 = L_6 = 0$ is a triple line and $L_1 = L_6 = 0$, $L_4 = L_5 = 0$ are double lines on the surface. The surface therefore also contains another system of conics; namely, those in the pencil of planes through the triple line.

The system has eight pinchpoints: the four intersections of $L_5 = L_6 = 0$ with the envelope of the system of planes, the two points $L_1 = L_6 = 3 L_3^2 - 4 L_2 L_4 = 0$ and the two points $L_4 = L_5 = 3 L_2^2 - 4 L_1 L_3 = 0$. It can not envelope a component of the nodal curve for the surface enveloped by the planes:

$$L_1 u^3 + 3 L_2 u^2 + 3 L_3 u + L_4 = 0$$

can not have a rectilinear directrix.

The other systems are those of the form:

$$L_1 u^3 + (L_2 - L_1) u^2 + (L_3 - L_4) u + L_4 = 0,$$
$$L_1 L_6 u^2 + [(L_2 + L_3) L_7 - L_1 L_6 - L_4 L_5] u + L_4 L_5 = 0.$$

By combining these two equations we obtain:

$$L_1 L_5 u^2 + (L_2 L_5 - L_1 L_5 - L_1 L_6) u + L_3 L_5 + L_1 L_6 - (L_2 + L_3) L_7 = 0.$$

At any point of the nodal curve the two quadratic equations have two common roots. The nodal curve is thus seen to be the intersection, other than $L_5 = L_6 = 0$ of

$$L_5 L_6 = L_5 L_7 + L_6 L_7,$$
$$L_3 L_5 L_6 + L_1 L_6^2 = L_4 L_5^2 + (L_2 + L_3) L_6 L_7.$$

It is a quintic curve with a triple point at $L_5 = L_6 = L_7 = 0$. This nodal quintic may decompose into a quartic with a node at $L_5 = L_6 = L_7 = 0$ and a straight through the node meeting the quartic again, or into a cubic and two straight lines both meeting the cubic at the same point and each meeting it again or into a conic and three concurrent lines each meeting the conic.

In each case where the curve is composite, a pencil of quadrics can be passed through the entire nodal curve exclusive of one nodal line. The residual intersections of such a system of quadrics with the surface is a system of conics whose planes envelope a quadric cone.

The system has eight pinchpoints. One conic of the system may be coplanar with the conic consecutive to it. The developable of the planes of the conics must then degenerate into a cone of third class.

The conics can not envelope the nodal curve. To see this first consider the case of a proper quintic. The conics can not be doubly tangent to this quintic, for if they were, each generator of the envelope of the planes of the conics would have to be a bisecant of the quintic. This is impossible, for of the two intersections of each generator of that developable with the cone,

$$L_5 L_6 = L_5 L_7 + L_6 L_7,$$

on which the quintic lies, at least one must lie on the residual curve of intersection of the surfaces. It follows that the conics do not envelope the quintic at all. For, the two conics through an arbitrary point of the quintic would be consecutive, yet each conic would have to meet on the quintic two conics not consecutive with it. When the quintic breaks up into a quartic and a line, a similar proof holds for the quartic. The line can not be enveloped since it is met by each conic but once.

When the quintic breaks up into a cubic and two straight lines, we may take these lines, since they intersect, for fundamental lines and a point on the cubic for fundamental point in a quadratic-quadratic Cremona transformation which transforms the surface into a quartic surface and the system of conics into a system of conics. If the original system touched the cubic, the transformed system would touch the conic into which the cubic is transformed. When, however, the inverse transformation is performed on such a system of conics on a quartic, the planes of the conics are seen to envelope a surface of class two instead of class three as here supposed.

The case of a nodal conic may be disposed of like that of a nodal cubic. The nodal conic may, however, be a cusp locus on the surface. When this happens, all the conics of the system meet this conic in a fixed point and the conic is nodal because two conics of the system coincide with it throughout.

III. *Systems of genus greater than zero.*

Any algebraic system of conics is determined by three equations of the form:

$$L_1 + L_2 u + L_3 v + \dots\dots\dots = 0$$
$$Q_1 + Q_2 u + Q_3 v + \dots\dots\dots = 0$$
$$f_n(u, v) = 0$$

wherein u and v are parameters and $f_n(u, v)$ is a polynomial in u and v of degree n and with constant coefficients. To each point on $f_n = 0$, considered as a curve in the (u, v) plane corresponds a conic of the system and conversely. The genus of the system is, therefore, equal to that of $f_n = 0$. Since the genus of the system is supposed greater than zero, we must have $n \geqq 3$.

If the equation determining the planes of the conics is considered as the equation of a curve in the (u, v) plane which meets $f_n(u, v) = 0$ in m points and if the equation of the quadrics, similarly considered, determines a curve which meets $f_n = 0$ in m' points, then the order of the surface determined by the system is, in general, $2m + m'$.

For certain pairs of values of u and v the corresponding quadrics will be composite. When such a point (u, v) lies on $f_n = 0$, and when the corresponding plane coincides with a component of the quadric, then the equation of the plane is a factor of the equation of the surface. As in the unicursal systems, when a plane is a component of its corresponding quadric, the equation of the system of quadrics may frequently be reduced.

It is usually true that only one conic of the system lies in an arbitrary plane of its developable. We may then take two of the non-homogeneous point coordinates of the planes for u and v. Since the coordinates (α, β, γ) of the planes of the developable satisfy the equations:

$$\gamma = \frac{F_m(\alpha, \beta)}{F_{m-1}(\alpha, \beta)} \qquad f_n(\alpha, \beta) = 0$$

it is seen that the equation of the planes of the conics may, in this case, be written:

$$L_1 + L_2 u + L_3 v + \frac{F_m}{F_{m-1}}, \ L_4 = 0.$$

At an arbitrary point of the nodal curve of the surface determined by the system, there are two pairs of values of (u, v) which satisfy the three given equations. The nodal curve is, therefore, determined by the condition that the three equations determine two common points in the (u, v) plane.

The pinchpoints are determined by the condition that the three given curves touch at a common point. This is five conditions on the five quantities u, v and the ratios of the coordinates of a point. The system has, therefore, in general, a finite number of pinchpoints. When, at a point, a conic meets two consecutive ones, the corresponding three (u, v) curves have contact of the second order at a common point. A curve which is touched by all the conics of the system is therefore determined by the condition that the three (u, v) curves touch at a common point; and one which has contact of the second order with all the conics, by the condition that the three (u, v) curves have contact of the second order.

Additional nodal right lines, through an arbitrary point of which passes only one conic of the system, may arise as in the case of unicursal systems and under similar conditions.

When a point (u, v) is a double point of $f_n(u, v) = 0$, the corresponding conic is, in general, a double conic on the surface. Moreover, if the multiple point is a cusp, the conic is a cusp locus, if the multiple point is a tacnode two sheets of the surface touch along the conic and similarly for higher singularities. This is seen by putting:

$$u - u_0 = t^n \qquad v - v_0 = a_1 t + a_2 t^2 + \dots\dots\dots$$

and determining the form of the surface in the neighborhood of the conic $t = 0$.

When, as is the case in the surfaces in which we shall be interested, the first two of the three given equations are linear in u and v, it is obvious that, at any point of the nodal curve other than that determined by multiple points of $f_n = 0$ and the right lines mentioned above, these two lines in the (u, v) plane must coincide. The nodal curve is, therefore, determined by the equations:

$$\frac{L_1}{Q_1} = \frac{L_2}{Q_2} = \frac{L_3}{Q_3}.$$

It is, in general, of order seven and of multiplicity n on the surface.

a. *Quartic surfaces.*

The system of conics:

$$L_1 + u L_2 = 0$$
$$L_1^2 + v Q_2 = 0$$
$$(a_0 v^2 + b_0 v + c_0) u^4 + (a_1 v^2 + b_1 v) u^3 + (a_2 v^2 + b_2 v) u^2 + a_3 v^2 u + a_4 v^2 = 0$$

determines a quartic surface. The system is of genus one, and the surface has no nodal curve, unless $f_n = 0$ has an additional double point, in which case the genus of the system is reduced to zero. In each plane of the pencil:

$$L_1 + u\,L_2 = 0$$

lie two conics of the system which touch at each of the points:

$$L_1 = L_2 = Q_2 = 0.$$

These points are tacnodal points on the surface. For four values of u the coplanar conics are consecutive. These four values of u are determined by the four tangents to $f_n = 0$ which are parallel to the v axis.

On the Steiner surface are systems of conics of any genus whatever, lying in the planes of the developables of tangents to the surface.

b. *Quintic surfaces.*

The system of conics satisfying the equations:

$$\begin{gathered} L_1 + u\,L_2 = 0 \\ L_1^2 + v\,Q_2 = 0 \\ (a_0 v^2 + b_0 v + c_0)\,u^5 + (a_1 v^2 + b_1 v + c_1)\,u^4 + (a_2 v^2 + b_2 v)\,u^3 + (a_3 v^2 + b_3 v)\,u^2 \\ + a_4 v^2 u + a_5 v^2 = 0 \end{gathered}$$

lies on a quintic surface. Two conics of the system lie in each plane of the pencil:

$$L_1 + u\,L_2 = 0$$

and touch at each of the tacnodal points:

$$L_1 = L_2 = Q_2 = 0.$$

The system is, in general, of genus two, but the genus may reduce by the appearance of additional double points in $f_n(u, v) = 0$. The surface has no nodal curve when the system of conics is of genus two, but, with decreasing genus, it has one or two nodal conics. These nodal conics may be cuspidal, consecutive, etc.

The number of consecutive coplanar conics is, at most, six.

The system of conics

$$\begin{gathered} L_1 + u\,L_2 = 0 \\ L_1 L_3 + v\,Q_2 = 0 \\ (a_0 v^2 + b_0 v + c_0)\,u^3 + (a_1 v^2 + b_1 v + c_1)\,u^2 + (a_2 v^2 + b_2 v)\,u + a_3 v^2 = 0 \end{gathered}$$

is of genus one and lies on a quintic surface.

The conic $L_3 = Q_2 = 0$ is nodal, or, in particular, cuspidal, on the surface. The surface may have an additional nodal line. $L_2 = L_4 = 0$ is such a line when:

$$a_0 = 0 \qquad Q_2 \equiv L_4^2 + L_2 L_5 \qquad b_0 L_5 + a_1 L_3 \equiv \alpha L_2 + \beta L_4$$

where α and β are constants.

The two conics in an arbitrary plane

$$L_1 + u L_2 = 0$$

meet twice on the nodal conic and also at each of the points

$$L_1 = L_2 = Q_2 = 0.$$

Four conics of the system are coplanar with the conics consecutive to them.

When the plane determined by an arbitrary conic of the system does not contain another conic of the system, the planes of the conics envelope a cone of class three. Since the imposition on this cone of the condition of being unicursal is equivalent to bringing an additional nodal conic on the surface, it is seen that the nodal curve is of order three.

This nodal curve can not be a proper cubic, however, for the surface would then be rational as is seen by letting correspond to any point of it the point in which a fixed plane is pierced by the bisecant to the nodal cubic through the given point and conversely. Neither can the nodal curve be a proper conic and a line for such a surface is easily seen to be either rational or composite.

There do exist, however, quintics determined by systems of conics, whose nodal curve consists of three concurrent straight lines. Each conic of the system intersects each nodal line and, since it must meet four other conics of the system, passes through the vertex of the cone determined by the planes of the conics.

Taking two of the lines for fundamental lines, and the vertex of the cone for fundamental point in a quadratic-quadratic Cremona transformation, the surface is transformed into a ruled quartic of genus one, having the fundamental point for simple point. The tangent cone to the quartic at this point is of the form:

$$L_1 u + L_2 v + L_3 = 0$$
$$f_3(u, v) = 0.$$

It is easily seen that an infinite number of cones of class three exist which are tangent to the quartic and whose equations are of the form:

$$L_1' u + L_2' v + L_3' = 0,$$

wherein u and v are joined by the same cubic relation $f_3 = 0$.

Performing, now, the inverse transformation, we have the system of conics determined by:

$$\begin{gathered} L_1 u + L_2 v + L_3 = 0 \\ Q_1 v + Q_2 v + Q_3 = 0 \\ f_3(u, v) = 0. \end{gathered}$$

These equations can be still further specialized. When the surface is a quintic, there must be four pairs of values of u and v satisfying $f_3 = 0$ for which the first equation is a factor of the second. It is easily seen that three of these four points in the (u, v) plane are collinear. Hence, except in the particular cases in which some of them are consecutive, the equations of the system may be written:

$$\begin{gathered} L_1 u + L_2 v + L_3 = 0 \\ L_1 (L_4 + L_3) u + L_3 (L_4 - L_1) = 0 \\ a u^2 v + b u v^2 + c (u^2 - u) + d u v + e v^2 + f v = 0. \end{gathered}$$

The equation of the surface is:

$$\begin{gathered} a L_2 L_3^2 (L_1 - L_4)^2 - b L_1 L_3^2 (L_1 - L_4)(L_1 + L_3) + c L_2^2 L_4 (L_3 + L_4)(L_1 - L_4) \\ + d L_1 L_2 L_3 (L_1 - L_4)(L_3 + L_4) - e L_1^2 L_3 (L_1 + L_3)(L_3 + L_4) + f L_1^2 L_2 (L_3 + L_4)^2 = 0. \end{gathered}$$

The three nodal lines are:

$$\begin{gathered} L_1 = L_4 = 0 \\ L_3 = L_4 = 0 \\ L_1 + L_3 = L_3 + L_4 = 0. \end{gathered}$$

These lines may become consecutive.

All the conics of the system touch $L_2 = 0$ at $L_1 = L_2 = L_3 = 0$. On each nodal line are four pinchpoints of the system. The conics obviously can not envelope any of the nodal lines since they meet each line in only one point.

U. S. NAVAL ACADEMY, *June* 10, 1906.